

# COMPUTING DULAC'S MAPS OF (ALMOST EVERY) NON-DEGENERATE SINGULARITIES

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**ABSTRACT.** We study the complex Dulac map for a holomorphic foliation near a non-degenerate singularity (both eigenvalues of the linearization are nonzero). We describe the order of magnitude of the first two terms of the asymptotic expansion and show how to compute explicitly those terms using characteristics supported in the leaves of the linearized foliation. We perform similarly the study of the Dulac time spent around the singularity. These results are formulated in a unified framework taking no heed to the usual dynamical discrimination (*i.e.* no matter whether the singularity is formally orbitally linearizable or not and regardless of the arithmetics of the eigenvalues) provided the foliation has enough (*i.e.* two) separatrices.

## 1. INTRODUCTION

We consider a germ of a holomorphic vector field at the origin of  $\mathbb{C}^2$

$$A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y}$$

admitting an isolated, non-degenerate singularity at  $(0, 0)$ . In other words the origin is the only local zero of the vector field, and its linear part  $[\nabla A, \nabla B]$  at this point is a  $2 \times 2$  matrix with two nonzero eigenvalues, of ratio  $\lambda \in \mathbb{C}_{\neq 0}$ . We make the further assumption that the vector field admits the following decomposition in a convenient local analytic chart:

$$(X) \quad X_R = \lambda x \frac{\partial}{\partial x} + (1 + R) y \frac{\partial}{\partial y}, \quad R \in x^a \mathbb{C}\{x, y\}$$

for some positive integer  $a$  satisfying the relation

$$(a) \quad \Re \left( a + \frac{1}{\lambda} \right) \geq 0.$$

This setting encompasses almost all non-degenerate singularities, including every kind of saddle singularities ( $\lambda < 0$ ), as will be discussed at the end of the introduction.

Our study is carried out on a fixed polydisc  $\mathcal{U} = \rho\mathbb{D} \times r\mathbb{D}$  small enough for the relation

$$(R) \quad \sup |R(\mathcal{U})| < 1$$

to hold. We write  $\mathcal{F}_R$  the foliation of  $\mathcal{U}$  whose leaves are defined by the integral curves of  $X_R$ . This foliation admits two special leaves (called separatrices) each of whose adherence corresponds to a branch of  $\{xy = 0\}$ . We write  $\hat{\mathcal{U}} := \mathcal{U} \setminus \{xy = 0\}$ . Outside  $\{x = 0\}$  the foliation is transverse everywhere to the fibers of the fibration

$$\Pi : (x, y) \longmapsto x.$$

Being given  $(x_*, y_*) \notin \hat{\mathcal{U}}$  it is thus possible (under suitable assumptions that will be detailed later on) to lift through  $\Pi$  a path  $\gamma$  linking  $x$  to  $x_*$  in the foliation, starting from the point  $(x, y_*)$ . The arrival end-point of the lifted path defines uniquely a point  $(x_*, y_x) \in \Pi^{-1}(x_*)$ . This construction yields a locally analytic map from a sub-domain of

the transverse disc  $\{y = y_*\}$  into the transverse  $\{x = x_*\}$ , which is known as the **Dulac map**

$$\mathcal{D}_R : x \neq 0 \mapsto y_x$$

of  $X_R$  associated to  $(x_*, y_*)$ .

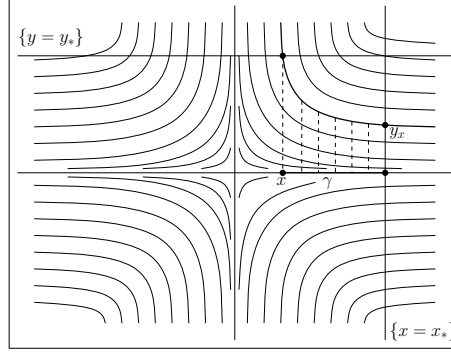


FIGURE 1.1. The Dulac map.

This map is in general multivalued, and its monodromy is generated by the holonomy of  $\mathcal{F}_R$  computed on  $\{x = x_*\}$  by winding around  $\{x = 0\}$ . The Dulac map governs part of the dynamics of  $\mathcal{F}_R$  and has been submitted to an intense study at least in the two settings we describe now.

- The proof of the Dulac conjecture regarding finiteness of the number of limit cycles for analytic vector fields in the real plane. Dulac maps are natural ingredients of the cross-section first-return map along a singular poly-cycle, whose attractive fixed-points correspond to limit cycles. As was noticed by Y. IL'YASHENKO [II'85] the original «proof» of H. DULAC [Dul23] crucially depends on a lemma which turned out to be false. Many powerful, if intricate, tools have been developed in the 1980's decade which finally led to a complete proof of Dulac's conjecture. Two parallel approaches evolved at the time to analyze the asymptotic expansion of the Dulac map: J. ÉCALLE [Éca92] studied it first formally using trans-series then through resurgent summation techniques, while Y. IL'YASHENKO [II'91] devised an argument based on super-accurate asymptotic series. Of course various authors contributed to the tale of Dulac's conjecture but the aim of this article is not to offer a comprehensive list.
- The study of the dynamics of a singular foliation through Seidenberg's reduction<sup>1</sup> process: Dulac maps appear naturally as «corner maps» encoding the transition between different components of the exceptional divisor. They measure how the different components of the projective holonomy pseudo-group mix together (see *e.g.* [Lor10] for a comprehensive presentation). In that context D. MARÍN and J.-F. MATTEI [MM08] proved recently that under suitable (generic) hypothesis a germ of a singular foliation is locally incompressible: there exists an adapted base of neighborhoods of  $(0, 0)$  in which the (non-trivial) cycles lying in the leaves of the restricted foliation must wind around the complement of the separatrix locus, in the trail of Milnor's theorem regarding holomorphic fibrations outside

<sup>1</sup>According to Seidenberg's algorithm (see [Sei68]) any isolated singularity of a germ of a holomorphic foliation  $\mathcal{F}$  can be «reduced» through a proper, rational map  $\pi : \mathcal{M} \rightarrow (\mathbb{C}^2, 0)$ , where  $\mathcal{M}$  is a conformal neighborhood of a tree  $E := \pi^{-1}(0, 0)$  of normally-crossing, conformal divisors  $\mathbb{P}_1(\mathbb{C})$ . The pulled-back foliation  $\pi^*\mathcal{F}$  has only isolated, reduced singularities (located on  $E$ ): either non-degenerate or of saddle-node type (exactly one nonzero eigenvalue).

the singular fibers. This study is the first step towards a complete analytical and topological classification of (generic) singular germs of a foliation. One of the main ingredients of their proof is to control the «roughness» of the corner maps and elements of the projective holonomy. This roughness can be read in the first two terms of the asymptotic expansion of the Dulac map.

The topic addressed here may appear worthless in view of the formidable work carried out by both J. ÉCALLE and Y. IL'YASHENKO. Yet it must be noticed that they were essentially busy with the real context and, although their techniques are naturally part of complex analysis, they focused on small neighborhoods of real trajectories. The originality of the present work lies in the study of the analytic continuation of the Dulac map on large sectors around the real directions. More precisely our results cover the following items, for saddle-like singularities (that is,  $\Re(\lambda) < 0$ ):

- we give a geometrical description of the maximal domain of definition (it contains germs of a sector around  $\{x = 0\}$  of arbitrary aperture),
- we express the Dulac map as an integral (more precisely, as the characteristics of the differential 1-form  $Rd(\log x^{-1/\lambda})$  along the leaves of  $\mathcal{F}_R$ ),
- we derive explicit bounds on the remainder of the asymptotic expansion in the sectors.

In particular it should be noted that although the Dulac map's asymptotic expansion can be expressed formally

$$\mathcal{D}_R(x) \simeq \sum_{n,m} D_{n,m} s_{n,m}(x) ,$$

where

$$s_{n,m}(x) = \begin{cases} \frac{x^{n\lambda+m}}{n\lambda+m} & \text{if } n\lambda + m \neq 0 \\ x^{n\lambda+m} \log x & \text{otherwise,} \end{cases}$$

the expansion converges if, and only if, the foliation is analytically normalizable<sup>2</sup>, as was proved by A. MOURTADA and R. MOUSSU [MM97, Proposition 1].

We also mention that the result exposed here allows to get rid of a non-necessary technical hypothesis in Marín-Mattei's theorem, namely discarding «bad<sup>3</sup>» irrational ratios in Seidenberg's reduction of the singularity. Proposition 1.5, stated further down, implies that the hypothesis of Lemma 4.4.3 of [MM08] are fulfilled, which was the only dependency in the chain of proof that needed to be addressed. The other restriction in their theorem, namely that no degenerate singularity occurs in the reduction process, is not technical and although it can be marginally weakened, must be kept (see [MT13] for examples of compressible foliations). Marín-Mattei's theorem now holds on the complement of a proper analytic set of the space of singular holomorphic foliations, the one avoiding degenerate singularities in the singularity's reduction.

**1.1. Statement of the main results.** The Dulac map is actually well-defined on the foliated universal covering of the restriction of  $\mathcal{F}_R$  to  $\hat{\mathcal{U}}$ , a complex surface isomorphic to the groupoid of paths tangent to  $\mathcal{F}_R$  with starting endpoint in  $\{y = y_*, x \neq 0\}$ . Our first result regards the structure of this space, which turns out to be embedded in the universal covering of  $\hat{\mathcal{U}}$ . This is another manner of saying that  $\mathcal{F}_R$  is incompressible in  $\mathcal{U}$ : every non-trivial cycle of a leaf of  $\mathcal{F}_R$  is a non-trivial cycle of  $\hat{\mathcal{U}}$ .

<sup>2</sup>Situation which arises not so often in the most interesting (quasi-)resonant cases.

<sup>3</sup>A «bad» irrational is unusually well approached by rational numbers. This only happens on a set of null Lebesgue-measure, but does nonetheless. Brjuno's explicit arithmetic condition [Brj71], expressed in terms of the convergents of  $\lambda$ , governs this behavior.

**Theorem 1.1.** *Let  $\mathcal{E} : (z, w) \mapsto (\exp z, \exp w)$  be the universal covering of  $\hat{\mathcal{U}}$ . Assume that conditions (X) and (R) are fulfilled. Then the foliation  $\mathcal{E}^*\mathcal{F}_R$  is regular and each one of its leaves is simply-connected.*

We understand now  $\mathcal{D}_R$  as a function  $z \mapsto w_z$ , after having fixed once and for all a preimage  $(z_*, w_*) \in \mathcal{E}^{-1}(x_*, y_*)$ . The domain of definition of  $\mathcal{D}_R$  corresponds to those  $(z, w_*)$  giving birth to a path  $\gamma_R(z)$  tangent to  $\mathcal{E}^*\mathcal{F}_R$  with landing endpoint  $(z_*, w_*)$ . We state now our main result.

**Theorem 1.2.** *Assume that conditions (X) and (R) hold.*

- (1)  $\mathcal{D}_R$  is holomorphic on a domain  $\Omega_* \ni z_*$  satisfying the following additional properties for any  $N \in \mathbb{N}_{>0}$ .

- If  $\Re(\lambda) \leq 0$  then  $\Omega_*$  is simply-connected and there exists  $r \geq r' > 0$  such that, for any  $w_*$  in  $\{\Re(w) < \ln r'\}$  and  $\Re(z_*) \leq \ln \rho$ , the domain  $\Omega_*$  contains at least the line segment

$$\{\Re(z) = \ln \rho, |\Im(z - z_*)| \leq \pi N\}$$

in its adherence. Besides  $\inf \Re(\Omega_* \cap \{|\Im(z - z_*)| \leq \pi N\}) > -\infty$ .

- If  $\Re(\lambda) < 0$  there exists  $0 < \rho' \leq \rho$  such that for every  $\Re(z_*) < \ln \rho'$  and  $N \in \mathbb{N}_{>0}$  the domain  $\Omega_*$  contains some infinite half-band

$$\{\Re(z) \leq \kappa', |\Im(z - z_*)| \leq \pi N\}$$

with  $\kappa' \leq \Re(z_*)$ .

- (2) For every  $z \in \Omega_*$  we have

$$\mathcal{D}_R(z) = w_* + \frac{z_* - z}{\lambda} + \frac{1}{\lambda} \int_{\gamma_R(z)} R \circ \mathcal{E} \, dz.$$

In particular  $\mathcal{D}_R(z_*) = w_*$ .

- (3) If  $\Re(\lambda) < 0$  and the condition (a) is satisfied then one has the asymptotic expansion

$$\int_{\gamma_R(z)} R \circ \mathcal{E} \, dz = \int_{\gamma_0(z)} R \circ \mathcal{E} \, dz + o(|z \exp^{-z/\lambda}|)$$

when  $\Re(z)$  tends to  $-\infty$  with a bounded imaginary part.

*Remark 1.3.* In (3) it only makes sense to consider that « $\Re(z)$  tends to  $-\infty$  with a bounded imaginary part» when  $\Re(\lambda) < 0$ , as underlined in (1). This is why we make the hypothesis on the real part of  $\lambda$  to precise the asymptotic behavior of the Dulac map. Although it is a technical artifact that can be overcome, by considering slanted bands instead of «horizontal» bands  $\{|\Im(z - z_*)| \leq \pi N\}$ , I surmise that  $\arg z$  must be somehow controlled in order to be able to obtain a useful asymptotic expansion as  $\Re(z) \rightarrow -\infty$ .

It is possible to carry out the computation of the characteristics associated to the model  $\mathcal{F}_0$ . The exact value of  $\int_{\gamma_0(z)} (x^n y^m) \circ \mathcal{E} dz$  does not offer an insightful interest as such. We can nonetheless deduce from it the dominant part of the characteristics  $\int_{\gamma_0(z)} R \circ \mathcal{E} dz$ , which splits into two components: the *regular part*  $\int_z^{z_*} R(\exp u, 0) du$ , which induces a holomorphic function in  $\mathcal{U}$  since  $R(0, 0) = 0$ , and the *resonant part* obtained by selecting in  $R$  only well-chosen monomials.

**Definition 1.4.** The **resonant support**  $\text{Res}(a, \lambda)$  associated to  $(a, \lambda)$  is

- the empty set if  $\lambda \notin \mathbb{R}_{<0}$ ,
- otherwise the subset of  $\mathbb{N}^2$  defined by

$$\text{Res}(a, \lambda) := \left\{ (n, m) \in \mathbb{N}^2 : m > 0, n \geq a, |n\lambda + m| < \frac{1}{2n} \right\}.$$

For  $G(x, y) = \sum_{n \geq 0, m \geq 0} G_{n,m} x^n y^m \in \mathbb{C}\{x, y\}$  we denote by  $G_0$  the **regular part** of  $G$

$$G_0(x) := G(x, 0)$$

and by  $G_{\text{Res}}$  its **resonant part**

$$G_{\text{Res}}(x, y) := \sum_{(n,m) \in \text{Res}(a, \lambda)} G_{n,m} x^n y^m.$$

It turns out that this support consists indeed of resonant or quasi-resonant monomials, according to the rationality of  $\lambda$  (Lemma 3.4), which carry without surprise the major part of the non-regular characteristics.

**Proposition 1.5.** *Assume that  $\Re(\lambda) < 0$  and condition (a) holds. Then for any  $G \in \mathbb{C}\{x, y\}$  we have*

$$\int_{\gamma_0(z)} G_0 \circ \mathcal{E} dz = G_0(0)(z_* - z) + O(\exp z)$$

and if moreover  $G \in x^a \mathbb{C}\{x, y\}$

$$\begin{aligned} \int_{\gamma_0(z)} G_{\text{Res}} \circ \mathcal{E} dz &= O(|z \exp^{-z/\lambda}|) \\ \int_{\gamma_0(z)} (G - G_{\text{Res}} - G_0) \circ \mathcal{E} dz &= O(|\exp^{-z/\lambda}|) \end{aligned}$$

(here again  $O(\bullet)$  regards the situation when  $\Re(z)$  tends to  $-\infty$  with a bounded imaginary part).

Notice that if  $\lambda < 0$  is irrational and  $G_{\text{Res}}$  is finitely supported then

$$\int_{\gamma_0(z)} G_{\text{Res}} \circ \mathcal{E} dz = O(|\exp^{-z/\lambda}|).$$

**1.2. Time spent near the singularity.** In applications (for instance in the study of real analytical vector fields of the plane) it is important to estimate the Dulac time, that is the time it takes to drift from  $(z, w_*)$  to  $(z_*, \mathcal{D}_R(z))$  in the flow of the vector field. In the case of  $X_R$  this time is obviously

$$\frac{z - z_*}{\lambda} = \int_{\gamma_R(z)} \frac{dz}{\lambda}.$$

Multiplying  $X_R$  by a holomorphic unit  $U$  does not change the underlying foliation (*i.e.* the Dulac map), although it does the Dulac time  $\mathcal{T}_{R,U}$ . The later is obtained by integrating a time-form<sup>4</sup>. P. MARDIŠIĆ, D. MARÍN and J. VILADELPRAT derived in [MMV08] the asymptotic expansion of the Dulac time for (deformations of) real planar vector fields when  $\lambda$  is rational. We wish to complete their study in the complex setting.

One can choose the time-form as

$$\tau(x, y) := \frac{dx}{\lambda x U(x, y)},$$

so that the next result holds.

**Theorem 1.6.** *Take  $U \in \mathcal{O}(\mathcal{U})$  such that  $U(0, 0) \neq 0$ . Assume that  $\mathcal{U}$  is chosen in such a way that, in addition to conditions (X) and (R), the holomorphic function  $U|_{\mathcal{U}}$  never vanishes. Then the Dulac time is holomorphic on  $\Omega_*$  and*

$$\mathcal{T}_{R,U}(z) = \int_{\gamma_R(z)} \frac{dz}{\lambda U \circ \mathcal{E}}.$$

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<sup>4</sup>A meromorphic 1-form  $\tau$  is a time-form for a vector field  $X$  when  $\tau(X) = 1$ .

If moreover  $\Re(\lambda) < 0$ , the condition (a) holds and  $U - U_0 \in x^a y \mathbb{C}\{x, y\}$  then, as  $\Re(z)$  tends to  $-\infty$  with bounded imaginary part,

$$\mathcal{T}_{R,U}(z) - \int_{\gamma_0(z)} \frac{dz}{\lambda U_0 \circ \mathcal{E}} = \int_{\gamma_0(z)} \frac{dz}{\lambda U_{\text{Res}} \circ \mathcal{E}} + o(|z \exp^{-z/\lambda}|).$$

The integral subtracted on the left-hand side boils down to  $\frac{z_* - z}{\lambda U(0,0)}$  if  $U - U(0,0)$  belongs to  $x^a y \mathbb{C}\{x, y\}$ . We mention that this situation can always be enforced by preparing the vector field:

**Lemma 1.7.** *Let  $Z$  be a germ of a holomorphic vector field near an isolated, non-degenerate singularity with ratio of eigenvalues  $\lambda \notin \mathbb{R}_{\geq 0}$ . Then there exists  $a \in \mathbb{N}_{>0}$  satisfying (a) and a choice of the local analytical coordinates such that  $Z = UX_R$  for two germs of a function satisfying  $R \in x^a y \mathbb{C}\{x, y\}$  and  $U - U(0,0) \in x^a y \mathbb{C}\{x, y\}$  with  $U(0,0) \neq 0$ .*

This lemma is plainly trivial when  $\lambda \notin \mathbb{R}_{\leq 0}$ : in that setting  $Z$  is locally analytically conjugate to its linear part, corresponding to  $R = 0$  and  $U = \text{cst}$ . When  $\lambda < 0$  is irrational the vector field is formally linearizable and can be put in the sought form for any finite order  $a \in \mathbb{N}$ , particularly one such that  $a + \frac{1}{\lambda} > 0$ . When  $\lambda = -\frac{p}{q}$  for  $p$  and  $q$  positive co-prime integers, resonances may appear and  $Z$  may not be even formally orbitally linearizable. These resonances correspond to pairs  $(n, m)$  of integers belonging to  $(q, p)\mathbb{N}$  (those for which  $n\lambda + m = 0$ ) in the Taylor expansion of  $R$  and  $U - U(0,0)$ , and as such cannot appear for an index  $n$  lesser than  $q$  or for  $m = 0$ . It is thus possible to cancel out formally the  $(q-1)$ -jet with respect to  $x$  and the 0-jet with respect to  $y$  of the given functions, meaning we can take  $a := q$ . Then  $a + \frac{1}{\lambda} \geq 0$ . The fact that this formal transform can always be chosen convergent is well known.

**1.3. Extension of the results to other singularities.** Condition (X) is satisfied except for some cases when  $\{\lambda, 1/\lambda\} \cap \mathbb{N} \neq \emptyset$ . The heuristic is that these singularities may not possess sufficiently many separatrices: the resonant node ( $\lambda \neq 0$ ) and the saddle-node ( $\lambda = 0$ , exactly one nonzero eigenvalue) admit only one in general. The former case is not very interesting since it corresponds to vector fields which can be analytically reduced to polynomial vector fields (Poincaré-Dulac normal forms) for which explicit computations are easily carried out. The geometry of the foliation itself is quite tame and completely understood. Save for some minor and technical complications, the framework we present can be adapted to encompass this case, although the trouble is not worth the induced lack of clarity in the exposition.

The case of the saddle-node is richer. In [MT13] we prove that Theorem 1.1 holds in that case. When the saddle-node is not divergent (*i.e.* it admits two separatrices) it can be brought in the form (X) and the Dulac map admits an integral representation as in Theorem 1.2 (2)

$$\mathcal{D}_R(z) = \mathcal{D}_{\mu x^k}(z) + \int_{\gamma_R(z)} (R - \mu x^k) \circ \mathcal{E} \frac{dz}{\exp(kz)}$$

where  $(k, \mu) \in \mathbb{N}_{>0} \times \mathbb{C}$  is the formal invariant of the saddle-node and  $\mathcal{D}_{\mu x^k}$  is the Dulac map for the normal form which can be explicitly computed:

$$\mathcal{D}_{\mu x^k}(z) = w_* + \mu(z - z_*) + \frac{\exp(-kz) - \exp(-kz_*)}{k}.$$

When the singularity is a divergent saddle-node it is possible to obtain an integral representation as well as a sectoral asymptotic behavior. We refer also to [Lor10] for more details.

After Seidenberg's reduction of its singularity a (germ of a) nilpotent foliation only has singular points either of non-degenerate or of saddle-node type. As a consequence the work done here and in [Lor10] is somehow sufficient to analyze more general Dulac maps. Yet there is a special case where it is not necessary to perform the reduction of singularities to be able to carry out some computations, which is in fact the most general formulation of the framework we introduce here, corresponding to vector fields in the form generalizing (X)

$$X_R = X_0 + RY$$

where:

- $X_0$  and  $Y$  are commuting, generically transverse vector fields,
- $Y$  admits a holomorphic first-integral  $u$  with connected fibers,
- $R \in u^a \mathbb{C} \{x, y\}$  for some  $a > 0$ .

Being given both a transverse disc  $\Sigma$  meeting a common separatrix of  $X_R$  and  $Y$  at some point  $p_*$ , and a transverse  $\Sigma'$  corresponding to a trajectory  $\{u = u_*\}$  of  $Y$ , we can define the Dulac map of  $X_R$  joining  $\Sigma$  to  $\Sigma'$  by lifting paths through the fibration  $(x, y) \mapsto u(x, y)$ . Then, with equality as multivalued maps on  $\Sigma \setminus \{p_*\}$ , we have the implicit relation

$$H_0 \circ \mathcal{D}_R = H_0 \circ \mathcal{D}_0 \times \exp \int_{\gamma_R} R \tau$$

where  $\tau$  is some time-form of  $X_R$  and  $H_0$  a first-integral<sup>5</sup> of  $X_0$ . With further work, and when applicable, it should be possible to derive the asymptotic behavior of  $\mathcal{D}_R$  near  $p_*$ , as is done here.

**1.4. Structure of the article.** This paper only uses elementary techniques and is consequently self-contained. Section 2 is devoted to the study of the geometry of the foliation and the analytic properties of the Dulac map. There are proved Theorem 1.1 and Theorem 1.2 (1) in the respective Section 2.2 and Section 2.3, while Theorem 1.2 (2) is established in Section 2.1. This paper ends with Section 3 where the explicit computation of characteristics  $\int_{\gamma_0(z)} G \circ \mathcal{E} dz$  are performed for the model  $\mathcal{F}_0$ . Yet the core of the section is the study of the asymptotic deviation between  $\int_{\gamma_R} G \circ \mathcal{E} dz$  and  $\int_{\gamma_0(z)} G \circ \mathcal{E} dz$ . Immediate consequences of this estimation are Theorem 1.2 (3) and the best part of Theorem 1.6. We end this paper with the proof of Proposition 1.5 in Section 3.2.3, completing Theorem 1.6.

## 2. ANALYTIC PROPERTIES OF THE DULAC MAP, GEOMETRY OF THE FOLIATION

We recall that  $\mathcal{U}$  is some polydisc centered at  $(0, 0)$  on which  $R$  is holomorphic and bounded, and we take  $(x_*, y_*) \in \hat{\mathcal{U}} := \mathcal{U} \setminus \{xy = 0\}$ . We will impose more technical assumptions on the size of  $\mathcal{U}$  in this section but they all shall be determined uniquely by

$$\|R\| := \sup_{\mathcal{U}} \left| \frac{R}{x^a} \right|.$$

We will mostly use the fact that if the supremum of  $|R|$  on  $\mathcal{U}$  is lesser than 1 then the foliation has a «controlled topological type».

**Definition 2.1.** All the paths  $\gamma$  we use throughout the paper are  $C^\infty$  maps from some compact interval  $\mathbb{I}$  into  $\mathcal{U}$ . Its starting point (*resp.* ending point) is written  $\gamma_*$  (*resp.*  $\gamma^*$ ).

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<sup>5</sup>This first-integral can be multivalued, as is the case considered above where  $H_0(x, y) = x^{-1/\lambda} y$ .



- (1) Let  $\Sigma \subset \mathcal{U} \setminus \{(0,0)\}$  be a cross-section transversal everywhere to  $\mathcal{F}_R$  (for short, a **transverse** to  $\mathcal{F}_R$ ). We introduce the groupoid  $\Gamma_R(\Sigma)$  of equivalence classes of paths  $\gamma$  tangent to  $\mathcal{F}_R$  with  $\gamma_* \in \Sigma$ , up to tangential homotopy (that is, homotopy within a given leaf of  $\mathcal{F}_R$ ) with fixed end-points. We call it **the tangential groupoid of  $\mathcal{F}_R$  relative to  $\Sigma$** .
- (2) The tangential groupoid of  $\mathcal{F}_R$  relative to  $\Sigma$  is naturally endowed with a structure of a foliated complex surface, which can be understood as the foliated universal covering of  $\text{Sat}_{\mathcal{F}_R}(\Sigma)$ , that is the locally biholomorphic, onto map

$$\begin{aligned} \sigma : \Gamma_R(\Sigma) &\longrightarrow \text{Sat}_{\mathcal{F}_R}(\Sigma) \\ \gamma &\longmapsto \gamma^*. \end{aligned}$$

- (3) The **Dulac map** of  $\mathcal{F}_R$  associated with  $(x_*, y_*)$  is the holomorphic function defined on

$$\Gamma_* : = \{ \gamma \in \Gamma_R(\mathcal{U} \cap \{y = y_*, x \neq 0\}) : \Pi(\gamma^*) = x_* \}$$

by

$$\begin{aligned} \mathcal{D}_R : \Gamma_* &\longrightarrow \Pi^{-1}(x_*) \\ \gamma &\longmapsto \gamma^*. \end{aligned}$$

### 2.1. Integral representation of the Dulac map.

**Definition 2.2.** A **time-form** of a vector field  $X$  is a meromorphic 1-form  $\tau$  such that  $\tau(X) = 1$ .

Because of the peculiar form of  $X_R$  one can always choose a time-form as

$$\tau := \frac{dx}{\lambda x}.$$

**Lemma 2.3.** Let  $\Sigma$  be a transverse to  $\mathcal{F}_R$ . For given  $G \in \mathcal{O}(\mathcal{U})$  the integration process

$$F : \gamma \in \Gamma_R(\Sigma) \longmapsto \int_{\gamma} G\tau$$

gives rise to a holomorphic function whose Lie derivative  $X_R \cdot F$  along  $X_R$  can be computed by considering  $F$  as a local analytic function of the end-point  $\gamma^*$ . Then

$$X_R \cdot F = G.$$

*Proof.* Outside the singular locus of  $X_R$  there exists a local rectifying system of coordinates: a one-to-one map  $\psi$  such that  $\psi^* X_R = \frac{\partial}{\partial t}$ . In these coordinates we have  $\psi^*(G\tau) = G \circ \psi dt$ . The fundamental theorem of integral calculus yields the result.  $\square$

Notice that  $X_0$  admits a (multivalued) first-integral with connected fibers

$$H_0(x, y) := x^{-1/\lambda} y,$$

which means that it lifts through  $\sigma$  to a holomorphic map, still written  $H_0$ , constant along the leaves of  $\mathcal{F}_0^\Sigma$  and whose range is in one-to-one correspondence with the space of leaves of  $\mathcal{F}_0^\Sigma$ .

**Lemma 2.4.** Let  $\Sigma$  be a transverse to  $\mathcal{F}_R$ . The function

$$\begin{aligned} H_R : \Gamma_R(\Sigma) &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto H_0 \exp \int_{\gamma} -R\tau \end{aligned}$$

is a holomorphic first-integral of  $\sigma^* \mathcal{F}_R$  with connected fibers.



*Proof.* The fact that  $H_R$  is holomorphic on  $\Gamma_R(\Sigma)$  is clear enough from Lemma 2.3. It is a first integral of  $\sigma^*\mathcal{F}_R$  if, and only if, the Lie derivative  $X_R \cdot H_R$  vanishes. This quantity is computed as follows:

$$\begin{aligned} X_R \cdot H_R &= X_0 \cdot H_R + R \left( y \frac{\partial}{\partial y} \right) \cdot H_R \\ &= H_R \times \left( X_R \cdot \int_{\gamma} -R\tau + R \frac{\left( y \frac{\partial}{\partial y} \right) \cdot H_0}{H_0} \right). \end{aligned}$$

Since  $\left( y \frac{\partial}{\partial y} \right) \cdot H_0 = H_0$  our claim holds. The fact that  $H_R$  has connected fibers is a direct consequence of both facts that  $H_0$  also has and  $H_R|_{\Sigma} = H_0$ .  $\square$

**Corollary 2.5.** *We have*

$$\mathcal{D}_R = \mathcal{D}_0 \times \exp \int_{\bullet} R\tau.$$

*Proof.* For any path  $\gamma \in \Gamma^*$  we have the relation  $H_R(\gamma_*) = H_R(\gamma)$ , that is

$$H_0(\gamma) \exp \int_{\gamma} -R\tau = H_0(\gamma_*).$$

The conclusion follows since  $\gamma \mapsto H_0(\gamma)$  is linear with respect to the  $y$ -coordinate of  $\gamma$  when  $x_*$  is fixed.  $\square$

**2.2. Foliated universal covering.** Proving Theorem 1.1 amounts to proving that the leaves of the foliation  $\tilde{\mathcal{F}} := \mathcal{E}^*\mathcal{F}_R$  are simply-connected. Write

$$\mathcal{E}^*X_R = \mathcal{E}^*X_0 + R \circ \mathcal{E} \times \mathcal{E}^* \left( y \frac{\partial}{\partial y} \right)$$

where

$$\begin{aligned} \mathcal{E}^*X_0 &= \lambda \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \\ \mathcal{E}^* \left( y \frac{\partial}{\partial y} \right) &= \frac{\partial}{\partial w}. \end{aligned}$$

The vector field  $\mathcal{E}^*X_R$  is holomorphic and regular on the complex rectangle

$$\tilde{\mathcal{U}} := \{ \Re(z) < \ln \rho, \Re(w) < \ln r \}.$$

It induces a foliation  $\tilde{\mathcal{F}}$  transversal to the fibers of

$$\Pi : (z, w) \mapsto z,$$

hence the leaf  $\tilde{\mathcal{L}}_{p_0}$  is everywhere locally the graph  $\{w = f(z)\}$  of some unique germ of a holomorphic function defined in a neighborhood  $V(p_0)$  of  $\Pi(p_0)$ . Because of this property the boundary of  $\tilde{\mathcal{L}}_{p_0}$  is included in the boundary of the domain of study

$$\partial \tilde{\mathcal{U}} = \{ \Re(z) = \ln \rho \text{ or } \Im(w) = \ln r \}.$$

The whole argument to come relies on the existence of a family of curves included in  $\tilde{\mathcal{L}}_{p_0}$  which project by  $\Pi$  on line segments of constant direction. A cycle  $\gamma$  within  $\tilde{\mathcal{L}}_{p_0}$  will therefore be pushed along those curves, as if repelled by the beam of a searchlight, resulting in a homotopy in  $\tilde{\mathcal{L}}_{p_0}$  with a path  $\hat{\gamma}$  whose projection bounds a region of empty interior. The compact image of  $\Pi \circ \hat{\gamma}$  can be covered by finitely many domains  $(V(p_n))_{n \leq d}$ , showing finally that  $\hat{\gamma}$  (and  $\gamma$ ) is homotopic in  $\tilde{\mathcal{L}}_{p_0}$  to a point.

**Definition 2.6.** (See Figure 2.1.) For  $v \in \{\Re(z) < \ln \rho\}$ ,  $0 < \delta < \pi$  and  $\vartheta \in \mathbb{S}^1$  the domain

$$S(v, \vartheta, \delta) := \{z : \Re(z) < \ln \rho, |\arg(z - v) - \arg \vartheta| < \delta\}$$

is called a **searchlight beam** of aperture  $2\delta$ , direction  $\vartheta$  and vertex  $v$ . If  $v = \Pi(p_0)$  we say it is a **stability beam** when the real part of the lift in  $\tilde{\mathcal{L}}_{p_0}$ , starting from  $p_0$ , of an outgoing ray  $t \geq 0 \mapsto v + t\theta$ , with  $|\arg \theta / \vartheta| < \delta$ , is decreasing.

*Remark 2.7.* This particularly means that the outgoing ray lifts *completely* in  $\tilde{\mathcal{L}}_{p_0}$  as long as it does not cross  $\{\Re(z) = \ln \rho\}$ .

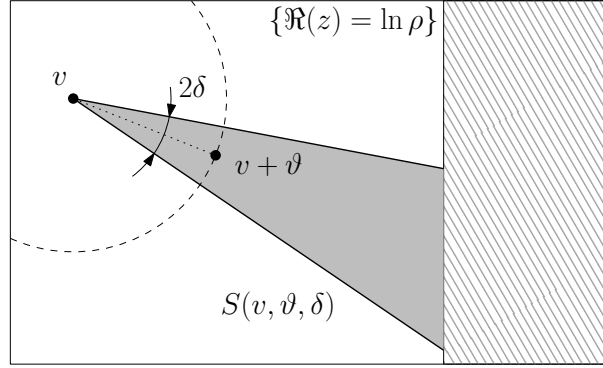


FIGURE 2.1. A searchlight beam.

The theorem is now a consequence of the next lemma:

**Lemma 2.8.** *There exists  $\delta \in ]0, \pi]$  and  $\vartheta \in \mathbb{S}^1$  such that for all  $p_0 = (z_0, w_0) \in \tilde{\mathcal{U}}$  the searchlight beam  $S(z_0, \vartheta, \delta)$  is a stability beam. We can take*

$$\begin{aligned} \vartheta &:= -\frac{\lambda}{|\lambda|} \\ \delta &\in ]0, \arccos(|R| \rho^a) [ , \end{aligned}$$

so that one can take  $\delta$  as close to  $\frac{\pi}{2}$  as one wishes by sufficiently diminishing  $\rho$ . Besides for any integral curve of  $\mathcal{E}^* X_R$  of the form  $t \mapsto (z_0 + t\theta, w(t))$  with  $t \geq 0$ ,  $\theta \in \mathbb{S}^1$  and  $w(0) = w_0$  we have the estimate for  $\Re(\theta) \neq 0$

$$(2.1) \quad \left| w(t) - w_0 - t \frac{\theta}{\lambda} \right| \leq \frac{\exp(a\Re(z_0))}{|\lambda \Re(\theta)|^a} \|R\| |1 - \exp(at\Re(\theta))|$$

and taking the limit  $\Re(\theta) \rightarrow 0$

$$(2.2) \quad \left| w(t) - w_0 \pm t \frac{i}{\lambda} \right| \leq \frac{\exp(a\Re(z_0))}{|\lambda|} t \|R\|$$

*Proof.* The lift in  $\tilde{\mathcal{F}}$  of a germ of a ray  $z(t) = z_0 + \theta t$ , with  $\theta \in \mathbb{S}^1$  and  $t \geq 0$ , starting from  $p_0$  is obtained as the solution to

$$\frac{\dot{w}}{\dot{z}}(t) = \frac{1 + R \circ \mathcal{E}(z(t), w(t))}{\lambda}, \quad w(0) = w_0,$$

that is

$$(2.3) \quad \dot{w}(t) = \frac{\theta}{\lambda} (1 + R \circ \mathcal{E}(z_0 + \theta t, w(t))).$$

The function  $t \mapsto \varphi(t) := \Re(w(t))$  is therefore solution to the differential equation

$$(2.4) \quad \dot{\varphi} = \Re \left( \frac{\theta}{\lambda} (1 + R \circ \mathcal{E} \circ (z, w)) \right),$$

which particularly means that

$$\left| \dot{\varphi}(t) - \Re \left( \frac{\theta}{\lambda} \right) \right| \leq \frac{\exp(a\Re(z_0))}{|\lambda|} \|R\| \exp(at\Re(\theta)) < \frac{\rho^a}{|\lambda|} \|R\|.$$

Exploiting the cruder estimate by taking  $\theta \in \vartheta \exp(i[-\delta, \delta])$  we derive

$$\dot{\varphi}(t) \leq \frac{1}{|\lambda|} (\cos \delta - \rho^a \|R\|) < 0.$$

Since  $\varphi(0) < \ln r$  it follows that  $\varphi(t) < \ln r$  as long as  $\Re((z(t))) < \ln \rho$ , which is our first claim.

Integrating both sides of the estimate yields

$$\left| \Re \left( w(t) - w_0 - t \frac{\theta}{\lambda} \right) \right| < \frac{\exp(a\Re(z_0))}{|\lambda \Re(\theta)| a} \|R\| |1 - \exp(at\Re(\theta))|.$$

The study we just performed can be carried out in just the same way for the imaginary part of  $w$ , yielding

$$\left| \Im \left( w(t) - w_0 - t \frac{\theta}{\lambda} \right) \right| < \frac{\exp(a\Re(z_0))}{|\lambda \Re(\theta)| a} \|R\| |1 - \exp(at\Re(\theta))|,$$

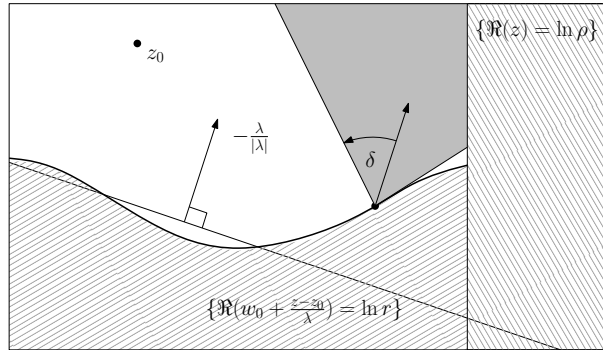
proving the sought estimate.  $\square$

An immediate consequence of the incompressibility of the foliation is the:

**Corollary 2.9.** *For any  $p_0 \in \tilde{\mathcal{U}}$  the leaf  $\tilde{\mathcal{L}}_{p_0}$  of  $\tilde{\mathcal{F}}$  passing through  $p_0$  is the graph of a holomorphic function on the simply-connected domain*

$$\Omega_{p_0} := \Pi \left( \tilde{\mathcal{L}}_{p_0} \right).$$

*Remark 2.10.* As a closing remark we should stress that the «roughness» of  $\partial\Omega_{p_0}$  is controlled by the aperture  $2\delta$  of the stability beam, which can be taken as close to  $\pi$  as one wishes, and by the direction  $\vartheta$  (which is that of the model). This is a kind of «conic-convexity» which forbids  $\partial\Omega_{p_0}$  to be too wild. In fact the closer  $\Re(z)$  is to  $-\infty$  the closer  $\Omega_{p_0}$  is to  $\left\{ \Re \left( w_0 + \frac{z-z_0}{\lambda} \right) < \ln r, \Re(z) < \ln \rho \right\}$  near  $z$ .



### 2.3. The domain of definition of the Dulac map : proof of Theorem 1.2 (1).

We continue to write  $\mathcal{D}_R$  for the Dulac map of  $\mathcal{F}_R$  associated to some couple  $(x_*, y_*)$  expressed in the coordinates  $\mathcal{E}$  (i.e. understood as a holomorphic function of  $(z, w)$ ). We recall that for any  $p_0 \in \tilde{\mathcal{U}}$  the leaf  $\tilde{\mathcal{L}}_{p_0}$  of  $\tilde{\mathcal{F}}$  passing through  $p_0$  projects on

$$\Omega_{p_0} := \Pi \left( \tilde{\mathcal{L}}_{p_0} \right).$$

**Proposition 2.11.** *We fix a preimage  $(z_*, w_*) \in \mathcal{E}^{-1}(x_*, y_*)$ .*

- (1) *The Dulac map is holomorphic on the open set*

$$\Omega := \left\{ z \in \mathbb{C} : (z, w_*) \in \tilde{\mathcal{U}} \text{ and } z_* \in \Omega_{(z, w_*)} \right\}.$$

*We write  $\Omega_*$  the connected component containing  $z_*$ .*

- (2) *If  $\Re(\lambda) \geq 0$  then  $\Omega = \Omega_*$  is a simply-connected domain. Particularly  $\text{adh}(\Omega_*) \cap \{\Re(z) = \ln \rho\}$  is a nonempty line segment. For every  $N \in \mathbb{N}_{>0}$  there exists  $r \geq r' > 0$  such that this line segment contains at least  $\ln \rho + i\Im(z_*) + [-\pi iN, \pi iN]$  for every  $\Re(w_*) < \ln r'$ .*
- (3) *If  $\Re(\lambda) < 0$  there exists  $0 < \rho' \leq \rho$  depending only on  $a, \lambda$  and  $\|R\|$  such that for every  $\Re(z_*) < \ln \rho'$  and  $N \in \mathbb{N}_{>0}$  the domain  $\Omega_*$  contains some infinite half-band  $\{\Re(z) \leq \kappa', |\Im(z - z_*)| \leq \pi N\}$  with  $\kappa' \leq \Re(z_*)$  depending only on  $N, a, \lambda$  and  $\|R\|$ .*

We first mention that  $\Omega$  is clearly open since if one can link a point  $(z, w_*)$  to  $(z_*, \mathcal{D}_R(z))$  with a compact tangent path  $\gamma$ , whose image is included in the open set  $\tilde{\mathcal{U}}$ , then surely this is again the case for a neighborhood of  $z$ . The rest of the section is devoted to proving the remaining items. In doing so we build an explicit tangent path linking  $(z_0, w_*)$  to  $(z_*, \mathcal{D}_R(z_0))$ , see Proposition 2.12 below, which will serve in the next section to establish the asymptotic expansion of the Dulac map through the integral formula of Corollary 2.5. We underline right now the fact that the projection  $\tilde{\gamma}$  of that path through  $\Pi$  does not depend on  $w_*$ , but only on  $z_0, a, \lambda, \rho, \|R\|$  and  $z_*$ .

**2.3.1. The integration path.** We write  $p_0 := (z_0, w_*)$ . If  $\Re(\lambda) \geq 0$  and  $z_0 \in \Omega$  then both stability beams  $S(z_0, \vartheta, \delta)$  and  $S(z_*, \vartheta, \delta)$  are included in  $\Omega_{p_0}$  and their intersection  $W$  is non-empty. Therefore  $z_0$  can be linked to  $z_*$  in  $\Omega_{p_0}$  by following first a ray segment of  $S(z_0, \vartheta, \delta)$  from  $z_0$  to some point  $z_1$  in  $W$ , then from this point backwards  $z_*$  along a ray segment of  $S(z_*, \vartheta, \delta)$ , as illustrated in Figure 2.2 below.



FIGURE 2.2. The path of integration  $\tilde{\gamma}$  when  $\Re(\lambda) \geq 0$ .

On the contrary if  $\Re(\lambda) < 0$  the candidate region  $W$  could be beyond  $\{\Re(z) = \ln \rho\}$ . The construction must therefore be adapted.

**Proposition 2.12.** *Assume  $\Re(\lambda) < 0$ . There exists  $\kappa \in \mathbb{R}$  depending only on  $a, \lambda, \|R\|$  and  $z_*$  for which the following property holds: for every  $z_0 \in \Omega_*$  one can choose a path  $\tilde{\gamma} : z_0 \rightarrow z_*$  with image inside  $\Omega_{p_0}$  in such a way that  $\tilde{\gamma}$  is a polygonal line of ordered vertexes  $(z_0, z_1, z_2, z_3, z_*)$  with (we refer also to Figure 2.3 below)*

- $z_1 = \max\{\kappa, \Re(z_0)\} + i\Im(z_0)$ ,
- $\arg(z_2 - z_1) = \arg \vartheta \pm \delta$ ,
- $\Re(z_2) = \Re(z_3) < \ln \rho$  and  $|\Im(z_3 - z_2)| \leq 2\pi N + \tan(|\arg \vartheta| + \delta)(\ln \rho - \kappa)$ ,
- $\arg(z_* - z_3) = \arg \vartheta \pm \delta$ .

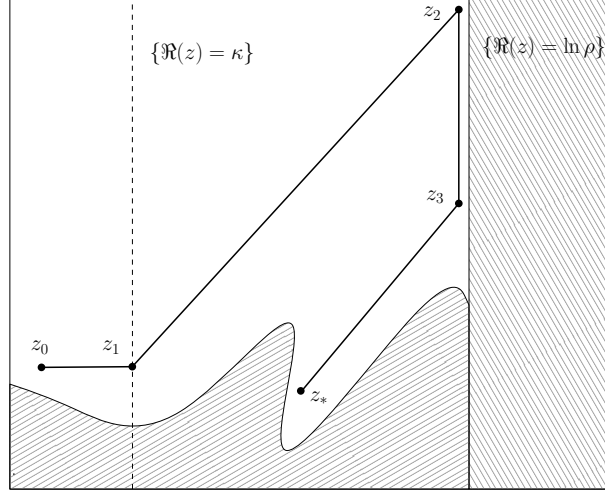


FIGURE 2.3. The path of integration  $\tilde{\gamma}$  when  $\Re(\lambda) < 0$ .

*Remark 2.13.* We could have made a similar construction without the use of  $\kappa$  (i.e. by joining directly  $z_0$  to some  $z_2$ ), but we need it in order to obtain uniform bounds with respect to  $\Im(z_0 - z_*)$  in the next section.

*Proof.* Write

$$\vartheta_{\pm} := \vartheta \exp(\pm i\delta).$$

Either  $\Re(\vartheta_+)$  if  $\Im(\lambda) \leq 0$  or  $\Re(\vartheta_-)$  if  $\Im(\lambda) \geq 0$  is positive, let us assume for the sake of example that  $\Im(\lambda) \leq 0$ , the other case being similar in every respect. There exists  $\kappa \leq \Re(z_*)$  such that  $\arg \vartheta = \arccos(\|R\| \exp(a\kappa))$ : in other words the ray segment  $(z_0 + \mathbb{R}_{\geq 0}) \cap \{\Re(z) < \kappa\}$  is included in  $\Omega_{p_0}$ . Obviously  $\kappa$  depends only on  $a, \lambda, \|R\|$  and  $z_*$ . We take for  $\tilde{\gamma}$  the polygonal line of ordered vertexes  $(z_0, z_1, z_2, z_3, z_*)$  built in the following fashion.

- If  $z_0 \in \{\Re(z) < \kappa\}$  then the partial ray  $(z_0 + \vartheta_0 \mathbb{R}_{\geq 0}) \cap \{\Re(z) < \kappa\}$ , included in  $\Omega_{p_0}$  according to Lemma 2.8, leaves the region at some point  $z_1$  with  $\Re(z_1) = \kappa$ . Otherwise we set  $z_1 := z_0$ .
- Both rays  $\{z_1, z_*\} + \vartheta_+ \mathbb{R}_{\geq 0}$ , included in  $\Omega_{p_0}$ , intersect the line  $\{\Re(z) = \ln \rho - \epsilon\}$  in, respectively,  $z_2$  and  $z_3$  for  $\epsilon > 0$  very small. The worst case scenario to bound  $|\Im(z_3 - z_2)|$  happens when  $z_1 = \kappa + i\Im(\pi N + z_*)$ , which gives the claimed estimate.

The line segment  $[z_2, z_3]$ , and therefore the whole image of  $\tilde{\gamma}$ , is included in  $\Omega_{p_0}$  thanks to the next lemma:

**Lemma 2.14.** *If  $\Re(\lambda) \leq 0$  then  $\text{adh}(\Omega_{p_0}) \cap \{\Re(z) = \ln \rho\}$  is a nonempty line segment. If  $\lambda \notin \mathbb{R}_{\geq 0}$  there exists  $\rho > 0$  such that the same property holds.*

*Proof.* In the case  $\Re(\lambda) \geq 0$  we have  $\max\{\Re(\vartheta \exp(\pm i\delta))\} > 0$ ; say, for the sake of example, that  $\vartheta_+ := \vartheta \exp(i\delta)$  has positive real part. If  $\lambda$  is not a positive number this property can be secured by decreasing  $\rho$  and taking  $\delta$  as close to  $\frac{\pi}{2}$  as need be. Take a path  $\Gamma$  connecting two points of  $\text{adh}(\Omega_{p_0}) \cap \{\Re(z) = \ln \rho\}$  (which is a non-empty set) within  $\Omega_{p_0}$ . Let  $I$  be the line segment of  $\{\Re(z) = \ln \rho\}$  joining those points. The ray  $p - \vartheta_+ \mathbb{R}_{\geq 0}$  emitted from some  $p \in I$  separates  $\{\Re(z) \leq \ln \rho\}$  into two connected regions. Since  $\Gamma$  starts from one of them and lands in the other one, the curve must cross the ray at some point  $q \in \Omega_{p_0}$ . The ray  $q + \vartheta_+ \mathbb{R}_{\geq 0}$  is included in  $\Omega_{p_0}$  since it lies within a stability beam, while it contains  $p$  in its adherence.  $\square$

### 2.3.2. The dual searchlight's sweep.

**Lemma 2.15.** *When  $\Re(\lambda) \geq 0$  the searchlight's beam  $S(z_0, -\vartheta, \delta)$  is included in  $\Omega$  for any  $z_0 \in \Omega$ . When  $\Re(\lambda) < 0$  the beam  $S(z_*, -\vartheta, \delta)$  is included in  $\Omega$ .*

For any  $z \in S(z_0, -\vartheta, \delta)$  we can link  $(z, w_*)$  to some point  $(z_0, w)$  with  $\Re(w) \leq \Re(w_*)$  by lifting in  $\tilde{\mathcal{F}}$  the line segment  $[z, z_0]$ . Therefore the lemma is trivial in the case where  $\Re(\lambda) < 0$ . On the contrary when  $\Re(\lambda) \geq 0$  the lemma is a consequence of the next one.

**Lemma 2.16.** *Assume that  $\Re(\lambda) \geq 0$ ,  $z_0 \in \Omega$  and let  $\eta := \Re(w_*)$ . Then for any other choice of  $w_*$  with real part lesser or equal to  $\eta$  we have  $z_0 \in \Omega$  as well.*

*Proof.* We set up a connectedness argument. Let  $B := \{w_* : \Re(w_*) \leq \eta\}$  and  $A := \{w_* : w_* \in B \text{ and } z_0 \in \Omega\}$ . By assumption  $A$  is not empty, and it is open in  $B$  for the same reason that  $\Omega$  is open. More precisely any  $w_* \in A$  admits a neighborhood  $V$  in  $B$  such that the image of  $\tilde{\gamma}$  is included in  $\Omega_{(z_0, w)}$  for every  $w \in V$ . Let now a sequence  $(w_n)_{n \in \mathbb{N}} \subset A$  converge towards  $w_\infty \in B$ . If  $\Re(\lambda) \geq 0$  then the image of  $\tilde{\gamma}$  is included in the union of the two stability beams  $S := S(z_0, \vartheta, \delta) \cup S(z_*, \vartheta, \delta)$  which are themselves included in every  $\Omega_{(z_0, w_n)}$ . Because the real analytic curves defining  $\partial\Omega_{(z_0, w)}$  vary continuously when  $w$  does we have  $S \subset \Omega_{(z_0, w_\infty)}$  also. In particular  $z_* \in \Omega_{(z_0, w_\infty)}$  and  $z_0$  belongs to  $\Omega_*$  for  $w_* := w_\infty$ . The former property implies in turn that  $A$  is a closed subset of  $B$  and as such spans the whole region  $B$ .  $\square$

*Remark 2.17.* When  $\Re(\lambda) < 0$  the above argument does not work since the image of  $\tilde{\gamma}$  must sometimes leave the (adherence of the) union of stability beams emitted by  $z_1$  and  $z_*$  (when it visits  $[z_2, z_3]$ ). Nothing guarantees that the limiting domain  $\Omega_{(z_0, w_\infty)}$  does not interrupt  $[z_2, z_3]$  at some point.

**2.3.3. Proof of (2).** The fact that  $\Omega$  is simply-connected is a consequence of Lemma 2.15, in the way that what was previously done to prove the simple-connectedness of  $\Omega_{p_0}$ . From this lemma also follows the fact that  $\text{adh}(\Omega) \cap \{\Re(z) = \ln \rho\}$  is a nonempty line segment, as can be seen by adapting in a straightforward way the proof of Lemma 2.14. In particular  $\Omega$  is connected.

To prove that  $\text{adh}(\Omega) \cap \{\Re(z) = \ln \rho\}$  can be arbitrarily big provided that  $\Re(w_*)$  be sufficiently small it is sufficient to invoke the fact that  $\{y = 0\}$  is the adherence of a separatrix of  $\mathcal{F}_R$ , so that  $\Gamma_R(\Sigma)$  contains elements winding more and more around  $\{x = 0\}$ .

**2.3.4. Proof of (3).** Because  $S(z_*, -\vartheta, \delta) \subset \Omega_*$  we only need to ensure that  $\Im(\vartheta_+)$  and  $\Im(\vartheta_-)$  have opposite signs. This can be enforced by taking  $\Re(z_*)$  negative enough, i.e. by taking  $\delta$  as close to  $\frac{\pi}{2}$  as need be.

## 3. ASYMPTOTICS OF THE DULAC MAP

Since  $\mathcal{D}_R$  is naturally defined on the universal covering of  $\mathcal{U} \setminus \{xy = 0\}$  we keep on working in logarithmic coordinates

$$(x, y) = \mathcal{E}(z, w) = (\exp z, \exp w) .$$

We fix once and for all a preimage  $(z_*, w_*) \in \mathcal{E}^{-1}(x_*, y_*)$ . For the sake of concision we make the convention that an object  $\mathcal{X}$  hatted with a *tilde* stands for its pull-back in logarithmic coordinates  $\tilde{\mathcal{X}} := \mathcal{E}^* \mathcal{X}$ . Notice that the time form  $\tau$  is transformed into

$$\tilde{\tau} = \frac{1}{\lambda} dz .$$

To obtain the image of  $x$  by the Dulac map  $\mathcal{D}_R$  we need to compute the integral

$$\int_{\gamma_R(x)} R\tau ,$$

where  $\gamma_R$  is a path tangent to  $\mathcal{F}_R$  linking  $(x, y_*)$  to some point of  $\Pi^{-1}(x_*)$ , and we intend more precisely to compare this value with that of

$$\int_{\gamma_0(x)} R\tau$$

which can be explicitly computed (Sub-section 3.2). The section's main result is the

**Theorem 3.1.** *Let  $N \in \mathbb{N}_{>0}$  be given. There exists a constant  $M > 0$  depending only on  $N, \lambda, a, \rho, \|R\|$  and  $\delta$  such that for any bounded  $G \in \mathcal{O}(\mathcal{U}) \cap x^a \mathbb{C}\{x, y\}$  and all  $x = \exp z$  with  $\Im(z - z_*) \leq \pi N$  one has*

$$\left| \int_{\gamma_R(x)} G\tau - \int_{\gamma_0(x)} G\tau \right| \leq M \left\| \frac{\partial G}{\partial y} \right\| |y_*| |x|^a .$$

*Remark 3.2.* Since  $\Re(a + \frac{1}{\lambda}) > 0$  we have  $|x|^a = o(|x^{-1/\lambda} \log x|)$ , proving the first part of Theorem 1.2 (3).

**3.1. Approximation to the formal model : proof of Theorem 3.1 (1).** We make here the hypothesis that  $\Re(\lambda) > 0$ . We need to compare this integral and the one obtained for the model, *i.e.* bound

$$\Delta(z_0) := \int_{\tilde{\gamma}(z_0)} \tilde{G}(z, w_R(z, z_0)) - \tilde{G}(z, w_0(z, z_0)) dz$$

where  $\tilde{\gamma}(z_0)$  is a path linking  $z_0$  to  $z_*$  within  $\Omega_{(z_0, w_*)}$  and  $z \mapsto w_R(z, z_0)$  is its lift in  $\tilde{\mathcal{F}}_R$  starting from  $(z_0, w_*)$ . We mention that

$$w_0(z, z_0) = w_* + \frac{z - z_0}{\lambda} .$$

For any  $(z, w_j) \in \tilde{\mathcal{U}}$  we have the estimate

$$\left| \tilde{G}(z, w_2) - \tilde{G}(z, w_1) \right| \leq |\exp(az)| \left\| \frac{\partial G}{\partial y} \right\| |\exp w_2 - \exp w_1|$$

so that

$$|\Delta(z_0)| \leq \left\| \frac{\partial G}{\partial y} \right\| \int_{\tilde{\gamma}} |\exp(az + w_0(z, z_0))| (\exp(w_R(z, z_0) - w_0(z, z_0)) - 1) dz .$$

Setting

$$D_R(z, z_0) := |w_R(z, z_0) - w_0(z, z_0)|$$



and taking  $|\exp z - 1| \leq |z| \exp |z|$  into account we derive

$$|\Delta(z_0)| \leq \left\| \frac{\partial G}{\partial y} \right\| \int_{\tilde{\gamma}} \exp \Re(a z + w_0(z, z_0)) D_R(z, z_0) \exp D_R(z, z_0) |dz| .$$

The proof is done when the next lemma is established:

**Lemma 3.3.** *There exists a constant  $K > 0$ , depending only on  $N, \lambda, a, \rho, \|R\|, z_*$  and  $\delta$ , such that*

$$\sup_t D_R(\tilde{\gamma}(t), z_0) \leq K$$

where  $\tilde{\gamma}$  is the integration path built in Proposition 2.12. The values of  $K$  is explicitly, if crudely, determined in the proof to come, and can surely be sharpened.

*Proof.* Invoking the estimate (2.1) from Lemma 2.8 and setting

$$\begin{aligned} C_1 &:= \frac{\|R\|}{a|\lambda|} \\ C_2 &:= \frac{C_1}{\Re(\vartheta_+)} \\ C_3 &:= a\rho^a C_1 \end{aligned}$$

we know that, using the number  $\kappa$  obtained in Proposition 2.12,

$$\begin{aligned} \sup_{z \in [z_0, z_1]} D_R(z, z_0) &\leq K_1 := C_1(\rho^a + \exp(a\kappa)) \\ \sup_{z \in [z_1, z_2]} D_R(z, z_0) &\leq K_2 := K_1 + C_2(\rho^a + \exp(a\kappa)) \\ \sup_{z \in [z_2, z_3]} D_R(z, z_0) &\leq K_3 := K_2 + C_3(2\pi N + \tan(|\arg \vartheta_+|)(\ln \rho - \kappa)) \\ \sup_{z \in [z_3, z_*]} D_R(z, z_0) &\leq K := K_3 + C_2(\rho^a + \exp \Re(a z_*)) . \end{aligned}$$

□

We conclude now the proof starting from

$$|\Delta(z_0)| \leq K \left\| \frac{\partial G}{\partial y} \right\| \exp \Re(K + w_* - z_0/\lambda) \int_{\tilde{\gamma}} \exp \Re\left(\left(a + \frac{1}{\lambda}\right)z\right) |dz| .$$

Let  $\alpha := |a + \frac{1}{\lambda}|$ . We bound each partial integral  $I_{\star \rightarrow \bullet} := \int_{[z_*, z_\bullet]} \exp \Re((a + \frac{1}{\lambda})z) |dz|$  in the following manner:

$$\begin{aligned}
I_{0 \rightarrow 1} &\leq \exp \Re \left( \left( a + \frac{1}{\lambda} \right) z_0 \right) \int_0^{\kappa - \Re(z_0)} \exp \left( t \Re \left( a + \frac{1}{\lambda} \right) \right) dt \\
&\leq \frac{\exp \Re \left( \left( a + \frac{1}{\lambda} \right) \kappa \right) - \exp \Re \left( \left( a + \frac{1}{\lambda} \right) z_0 \right)}{\Re \left( a + \frac{1}{\lambda} \right)}, \\
I_{1 \rightarrow 2} &\leq \exp \Re \left( \left( a + \frac{1}{\lambda} \right) z_1 \right) \int_0^{\ln \rho - \kappa} \exp \left( t \Re \left( \left( a + \frac{1}{\lambda} \right) \vartheta_+ \right) \right) dt \\
&\leq \exp(\alpha |z_1|) \frac{\exp(\alpha (\ln \rho - \kappa)) - 1}{\alpha} \\
&\leq \exp \left( \alpha \left( \sqrt{\kappa^2 + \pi^2 N^2} + \ln \rho - \kappa \right) \right), \\
I_{2 \rightarrow 3} &\leq \exp(\alpha |z_2|) \int_0^{\Im(z_3 - z_2)} \exp \left( t \left| \Im \left( \frac{1}{\lambda} \right) \right| \right) |dt| \\
&\leq \exp \left( \alpha |z_2| + \left| \frac{1}{\lambda} \right| |\Im(z_3 - z_2)| \right) \\
&\leq \exp(\alpha (|z_*| + \pi N + \tan(|\arg \vartheta| + \delta) (\ln \rho - \kappa))) \\
&\quad \times \exp \left( \left| \frac{1}{\lambda} \right| (2\pi N + \tan(|\arg \vartheta| + \delta) (\ln \rho - \kappa)) \right), \\
I_{3 \rightarrow *} &\leq \exp \left( \alpha \left( \sqrt{\Re(z_*)^2 + \pi^2 N^2} + \ln \rho - \Re(z_*) \right) \right).
\end{aligned}$$

In particular the dominant integral in the above list is  $I_{0 \rightarrow 1}$ , so that there exists a constant  $M$ , satisfying the required dependency properties, with

$$|\Delta(z_0)| \leq M \left\| \frac{\partial G}{\partial y} \right\| \exp \Re(w_* + az_0).$$

Since  $\Re(a + \frac{1}{\lambda}) > 0$  we have

$$\Delta(z_0) = o \left( |z_0| \exp \Re \left( \frac{-z_0}{\lambda} \right) \right)$$

as expected.

### 3.2. Study of the model.

3.2.1. *Explicit computation.* We want to compute for  $n, m \in \mathbb{N}$  the functions defined by

$$\begin{aligned}
T_{n,m}(z) &:= \int_{\tilde{\gamma}(z)} \exp(nu + mw_0(u, z_0)) du \\
&= \exp \left( m \left( w_* - \frac{z}{\lambda} \right) \right) \times \int_z^{z_*} \exp \left( \left( n + \frac{m}{\lambda} \right) u \right) du.
\end{aligned}$$

If  $n + m/\lambda = 0$  then  $\lambda = -p/q$ , with  $p$  and  $q$  co-prime positive integers, and  $(n, m) = k(q, p)$  with  $k \in \mathbb{N}$ . In that case, and when  $k > 0$ ,

$$(3.1) \quad T_{kq, kp}(z) = (z_* - z) \exp(k(pw_* + qz)) = O(|z \exp \Re(az)|).$$

The other case  $n + m/\lambda \neq 0$  is not more difficult:

$$T_{n,m}(z) = \exp(mw_* + nz) \frac{\exp((n + m/\lambda)(z_* - z)) - 1}{n + m/\lambda}.$$

One can see easily that as  $n + m/\lambda$  tends to zero (which may happen if, and only if,  $\lambda$  is a negative irrational) the function  $T_{n,m}$  grows in modulus. The dominant support introduced in Definition 1.4 allows to discriminate between two kind of growth rate.

3.2.2. *Resonant support.* We show now that the resonant support consists of (quasi-)resonant monomials only.

**Lemma 3.4.** *Assume that  $\lambda < 0$  and  $a + \frac{1}{\lambda} > 0$ .*

(1) *If  $\lambda = -p/q < 0$  is a rational number then*

$$\text{Res}(a, \lambda) = \{k(q, p) : k \in \mathbb{N}, kq \geq a\}.$$

(2) *If  $\lambda$  is a negative irrational we denote by  $(-p_k/q_k)_{k \in \mathbb{N}}$  its sequence of convergents. Then*

$$\text{Res}(a, \lambda) = \{(q_k, p_k) : k \in \mathbb{N}, q_k \geq a\}.$$

*Proof.*

(1) Because we have  $n \geq a \geq q$  the relation  $|n\lambda + m| < \frac{1}{2n}$  becomes

$$|np - mq| < \frac{q}{2n} < 1.$$

Hence  $np = mq$  and since  $p$  and  $q$  are co-prime the conclusion follows.

(2) This is a consequence of the well-known result in continued-fraction theory: if  $\frac{p}{q} \in \mathbb{Q}_{>0}$  is given such that  $\left|\frac{p}{q} + \lambda\right| < q^{-2}/2$  then  $(p, q)$  is one of the convergents of  $|\lambda|$ . □

3.2.3. *Dominant terms: proof of Proposition 1.5.* Nothing needs to be proved for  $G_0$  so we assume that  $G$  expands into a power series  $G(x, y) = \sum_{n \geq a, m > 0} G_{n,m} x^n y^m$  convergent on a closed polydisc of poly-radii at least  $(\rho + \epsilon, r + \epsilon)$ . Because of the Cauchy formula, for all  $n, m$

$$|G_{n,m}| \leq C(\rho + \epsilon)^{-n} (r + \epsilon)^{-m}$$

where  $C := \sup_{|x|=\rho+\epsilon, |y|=r+\epsilon} |G(x, y)|$ .

If  $\lambda$  is not real then

$$\inf_{(n,m) \in \mathbb{N}^2 \setminus \{(0,0)\}} |n\lambda + m| \geq a |\Im(\lambda)|.$$

Let  $z - z_*$  be given with imaginary part bounded by  $N\pi$  for some integer  $N > 0$  and with real part lesser than

$$\mu := - \left| \frac{\Im(\lambda)}{\Re(\lambda)} \right| N\pi.$$

By construction of  $\mu$  we have

$$\begin{aligned} \Re\left(\frac{m}{\lambda}(z_* - z)\right) &= \frac{m}{|\lambda|^2} (\Re(\lambda) \Re(z_* - z) + \Im(\lambda) \Im(z_* - z)) < 0 \\ &\leq \Re\left(\frac{1}{\lambda}(z_* - z)\right) \end{aligned}$$

so that we derive at last

$$|T_{n,m}(z)| \leq \frac{2|\lambda| r^m \rho^{n+\Re(1/\lambda)}}{a |\Im(\lambda)|} \exp \Re\left(-\frac{z}{\lambda}\right).$$

Now

$$\begin{aligned} \left| \int_z^{z_*} G \circ \mathcal{E} dz \right| &= \left| \sum_{n \geq a, m > 0} G_{n,m} T_{n,m}(z) \right| \\ &\leq \frac{2|\lambda| \rho^{\Re(1/\lambda)} C}{a |\Im(\lambda)|} \times \frac{(r+\epsilon)(\rho+\epsilon)}{\epsilon^2} \times \exp \Re\left(-\frac{z}{\lambda}\right) \\ &= O(|\exp -z/\lambda|), \end{aligned}$$

ending the proof for the non-real case. In fact this reasoning goes on holding even when  $\lambda < 0$  as long as  $(n, m)$  belongs not to  $\mathbf{Res}(a, \lambda)$ , since in that case

$$|n + m/\lambda| > \frac{|\lambda|}{2n}$$

has strictly sub-geometric inverse.

Take now  $\lambda$  negative real and  $G = G_{\mathbf{Res}}$ . If  $\lambda = -p/q$  is a negative rational number then (3.1) provides what remains to be proved. Assume now that  $\lambda$  is irrational. Because  $|\exp z - 1| \leq |z|$  when  $\Re(z) < 0$  we have for  $(n, m) \in \mathbf{Res}(a, \lambda)$

$$|T_{n,m}(z)| \leq |(z - z_*) \exp(nz)| r^m.$$

Therefore

$$\begin{aligned} \left| \int_{\gamma_0(z)} G_{\mathbf{Res}} \circ \mathcal{E} dz \right| &\leq C \sum_{(n,m) \in \mathbf{Res}(a,\lambda)} (\rho+\epsilon)^{-n} (r+\epsilon)^{-m} |T_{n,m}(z)| \\ &\leq C |z_* - z| \sum_{(n,m) \in \mathbf{Res}(a,\lambda)} \left( \frac{\exp \Re(z)}{\rho+\epsilon} \right)^n \left( \frac{r}{r+\epsilon} \right)^m \\ &\leq C \frac{(r+\epsilon)(\rho+\epsilon)}{\epsilon^2} |z_* - z| \exp \Re(az) \\ &= O(|z \exp -z/\lambda|) \end{aligned}$$

as expected.

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